Some solutions to Midterm Exams.

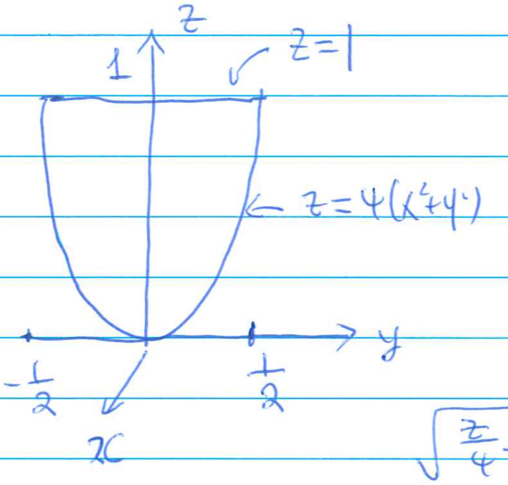
Version I

(6) Consider

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}} \int_{4(x^2+y^2)}^{1} f(x,y,z)\,dz\,dy\,dx.$$

Put it in $dx\,dy\,dz$.

$\boxed{\text{Sol}}$



$z = 4(x^2+y^2) \iff x = \pm\sqrt{\frac{z}{4}-y^2}$

$$\therefore \quad I = \iint_D \int_{-\sqrt{\frac{z}{4}-y^2}}^{\sqrt{\frac{z}{4}-y^2}} f(x,y,z)\,dx\,dA(y,z), \text{ where}$$

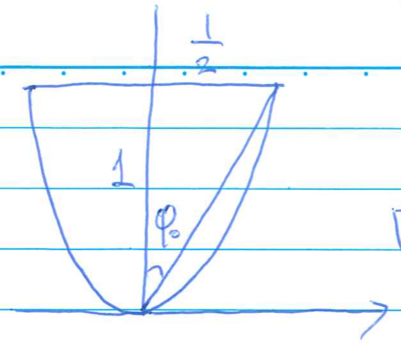
$$D = \left\{ (y,z) : -\sqrt{\tfrac{z}{4}} \le y \le \sqrt{\tfrac{z}{4}},\ 0 \le z \le 1 \right\}$$

$$\therefore \quad I = \int_0^1 \int_{-\sqrt{\frac{z}{4}}}^{\sqrt{\frac{z}{4}}} \int_{-\sqrt{\frac{z}{4}-y^2}}^{\sqrt{\frac{z}{4}-y^2}} f(x,y,z)\,dx\,dy\,dz.$$

(7) I as in (6). Order in Polar coordinates

$\downarrow$ (P.T.O.)

$\boxed{\text{Sol.}}$



$$\tan\varphi_0 = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\varphi_0 = \tan^{-1}\frac{1}{2} \in (0, \pi).$$

For φ, $0 \leq \varphi \leq \varphi_0$, every ray from the origin hits $z=1$, ie, $\rho = 1/\cos\varphi$, so the integral in this part is

$$\int_0^{2\pi}\int_0^{\varphi_0}\int_0^{\frac{1}{\cos\varphi}} f(\cdot,\cdot,\cdot)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

For φ, $\varphi_0 \leq \varphi \leq \pi/2$, every ray hits $z = 4(x^2+y^2)$. In polar,

$$z = 4(x^2+y^2) \iff$$

$$\rho\cos\varphi = 4\rho^2\sin^2\varphi, \iff$$

$$\rho = \frac{\cos\varphi}{4\sin^2\varphi}$$

$\therefore$ the integral in this part =

$$\int_0^{2\pi}\int_{\varphi_0}^{\pi/2}\int_0^{\frac{\cos\varphi}{4\sin^2\varphi}} f(\cdot,\cdot,\cdot)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

$$\therefore\ I = \left(\int_0^{2\pi}\int_0^{\varphi_0}\int_0^{\frac{1}{\cos\varphi}} + \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\frac{\cos\varphi}{4\sin^2\varphi}}\right) f(\rho\cos\theta\sin\varphi,$$

$$\rho\sin\theta\sin\varphi, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

(8) Let $u = x+y-z = \pm 1$, $v = x-2y = \pm 2$,
$w = x+z = \pm\pi/2$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = -2-1-2 = -5$$

$$\therefore \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|} = \frac{1}{5}.$$

Using change of variables formula,

$$\iiint_P \cos(x+z)\,dV(x,y,z)$$

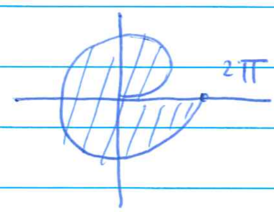
$$= \int_{-1}^{1}\int_{-2}^{2}\int_{-\pi/2}^{\pi/2} \cos w \times \frac{1}{5}\,dw\,dv\,du$$

$$= \frac{1}{5}\times 2\times 4\times \sin w\Big|_{-\pi/2}^{\pi/2} = \frac{16}{5} \;\#$$

Version II

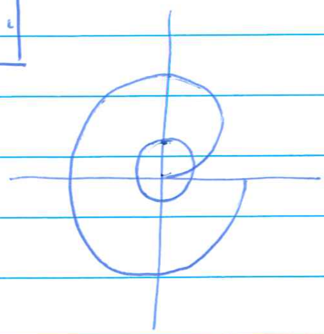
(5) S region bdd by $r=\theta$, $\theta\in[0,2\pi]$, and $[0,2\pi]$ on x-axis. Find the improper integral

$$\iint_S \log(x^2+y^2)\,dA(x,y).$$



Sol. $S = \{(r,\theta) : \theta\in[0,2\pi],\ 0\le r\le\theta\}$

Sol.



$r = \theta$ here

$$S_\varepsilon = S \setminus \{(x,y) : x^2+y^2 \le \varepsilon^2\}$$

$$I_\varepsilon = \iint_{S_3} \log(x^2+y^2)\, dA(x,y)$$

$$= \int_0^{2\pi} \int_\varepsilon^\theta (\log r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \theta^2(\log\theta^2 - 1) - \varepsilon^2(\log\varepsilon^2 - 1)\, d\theta$$

Using $\varepsilon \log \varepsilon \to 0$ as $\varepsilon \to 0$, we see that

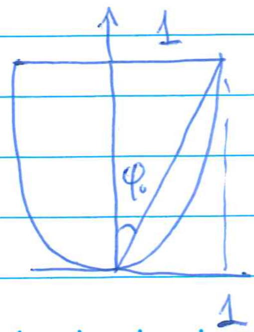
$$I_\varepsilon \to \int_0^{2\pi} \theta^2(\log\theta^2 - 1)\, d\theta \text{ , ie}$$

the improper integral exists and is equal to

$$\int_0^{2\pi} \theta^2(\log\theta^2 - 1)\, d\theta.$$

Using $\int$ by parts you could evalute this integral.

(6) similar to Problem (6) in Version I (just 1/4 replaced by 1)



$$\tan\varphi_0 = 1/1 = 1 \Rightarrow \varphi_0 = \pi/4.$$

$$\therefore I = \int_0^1 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(\cdot,\cdot,\cdot)\, dx\, dy\, dz.$$

⑦ Just like ⑦ : Version I, $\varphi_0 = \pi/4$ now.

⑧ $x+y-z=\pm 1$, $x-2y=\pm \pi/2$, $x+z=\pm 3$. Find

$$\iiint_P \cos(x-2y)\, dV.$$

Let
$$u = x+y-z \in [-1,1]$$
$$v = x-2y \in [-\pi/2, \pi/2]$$
$$w = x+z \in [-3,3]$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = -5$$

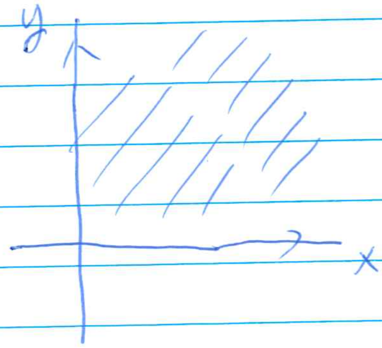
$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{5}$$

$$\therefore\ I = \int_{-1}^{1}\int_{-3}^{3}\int_{-\pi/2}^{\pi/2} \cos v \,\frac{1}{5}\, dv\, dw\, du$$

$$= \frac{1}{5}\sin v\Big|_{-\pi/2}^{\pi/2} \times 6 \times 2 = \frac{24}{5}. \#$$

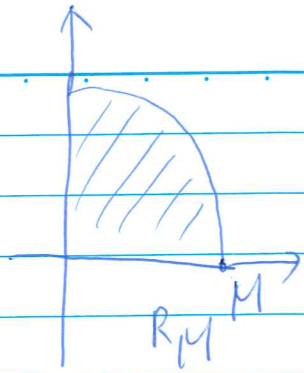
Version III

⑤ $$\iint_R \frac{dA}{(1+x^2+y^2)^\alpha \log(x^2+y^2)},\quad R$$



Sol.

Let $R_M = \{(x,y) : x^2+y^2 \le M^2,\ x,y \ge 0\}$



$$I_M = \iint_{R_M} \frac{dA}{(1+x^2+y^2)^\alpha \log(x^2+y^2)}, \quad \alpha > 0.$$

$$= \int_0^{\pi/2} \int_0^M \frac{r\,dr\,d\theta}{(1+r^2)^\alpha \log r^2}$$

~~if I_M has a limit when $M \to \infty$, then the improper~~

integral exists. When M is v. large,

$$\frac{r}{(1+r^2)^\alpha \log r^2} \sim \frac{r}{r^{2\alpha} \log r^2} = \frac{r^{1-2\alpha}}{\log r^2}$$

$$\int_M^L \frac{r\,dr}{(1+r^2)^\alpha \log r^2} \le \int_M^L \frac{r^{1-2\alpha}}{\log r^2}\,dr \le \int_M^L r^{1-2\alpha}\,dr$$

$$= \frac{1}{2-2\alpha} r^{2-2\alpha} \Big]_M^L .$$

When $2-2\alpha < 0$, ie, $1 < \alpha$, this integral $\to 0$ as $L, M \to \infty$.

$\therefore$ The improper integral exists when $\alpha > 1$.

When $2-2\alpha > 0$, ie, $\alpha < 1$, the integral $\to \infty$ as $L \to \infty$.

$\therefore$ The improper integral not exist.

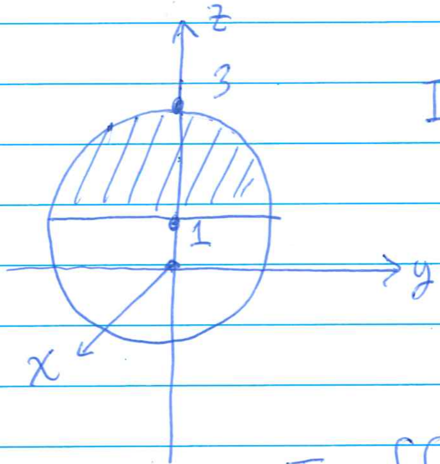
At $\alpha = 1$, $\dfrac{r}{(1+r^2)^\alpha \log r^2} \sim \dfrac{1}{2r\log r}$

$$\int_M^L \frac{dr}{2r\log r} = \frac{1}{2}\log(\log r)\Big|_M^L \to \infty \quad \text{as } L \to \infty$$

$\therefore$ when $\alpha = 1$, the improper integral not exist.

Conclusion: $\alpha \in (-\infty, 1)$ is the range for the existence of the improper integral.

(6)



$$I = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{1}^{\sqrt{4-x^2-y^2}+1} f\,dz\,dy\,dz$$

Convert to $dx\,dy\,dz$

$$I = \iint_D \int_{-\sqrt{4-y^2-(z-1)^2}}^{\sqrt{4-y^2-(z-1)^2}} f(x,y,z)\,dx\,dA(y,z)$$

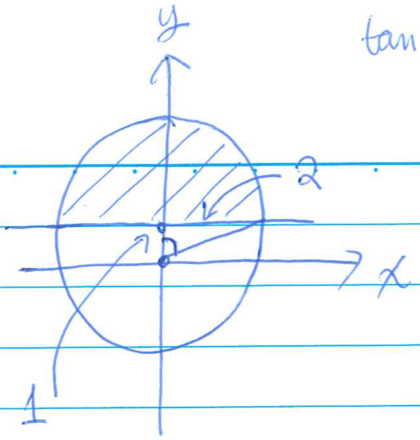
$z = \sqrt{4-x^2-y^2}+1$

$x = \pm\sqrt{4-y^2-(z-1)^2}$

$$D = \{(y,z) : 1 \le z \le 3,\ -\sqrt{4-(z-1)^2} \le y \le \sqrt{4-(z-1)^2}\}$$

$$\therefore I = \int_1^3 \int_{-\sqrt{4-(z-1)^2}}^{\sqrt{4-(z-1)^2}} \int_{-\sqrt{4-y^2-(z-1)^2}}^{\sqrt{4-y^2-(z-1)^2}} f(x,y,z)\,dx\,dy\,dz.$$

(7) Convert I in (6) to polar coordinates.



$\tan\varphi_0 = \frac{2}{1} = 2, \quad \varphi_0 = \tan^{-1} 2 \in (0,\pi)$

Every ray from the origin first hits $z=1$ and then hit $z=\sqrt{4-x^2-y^2}+1$

$z=1 \iff \rho = 1/\cos\varphi$

$z=\sqrt{4-x^2-y^2}+1 \iff \rho\cos\varphi = \sqrt{4-\rho^2\sin^2\varphi}+1$, ie

$$\rho = \cos\varphi + \sqrt{\cos^2\varphi+3}$$

$$\therefore \quad I = \int_0^{2\pi}\int_0^{\varphi_0}\int_{1/\cos\varphi}^{\cos\varphi+\sqrt{\cos^2\varphi+3}} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta.$$

(8) $x+y-z=\pm 1$, $x-2y=\pm 2$, $x+z=\pm\pi/2$ form a parallelotope P.

$$\iiint_P \sin^2(x+z)\,dV.$$

$\boxed{\text{Sol.}}$

$u = x+y-z \in [-1,1]$

$v = x-2y \in [-2,2]$

$w = x+z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = -5$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{5}.$$

$$I = \int_{-2}^{2}\int_{-1}^{1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 v \times \frac{1}{5} \times dv\,du\,dw$$

$$= 4 \times 2 \times \frac{1}{5} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2v)\,dv$$

$$= \frac{4}{5}\left(v - \frac{1}{2}\sin 2v\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{4\pi}{5} \,.\, \#$$